## § 4.3 Continuous functions

Let $f: D \subset \mathbb{R} \to \mathbb{R}$.

**Definition 4.4:**

$f$ is "continuous on $D$", if $f$ is continuous at each point $x_0 \in D$.

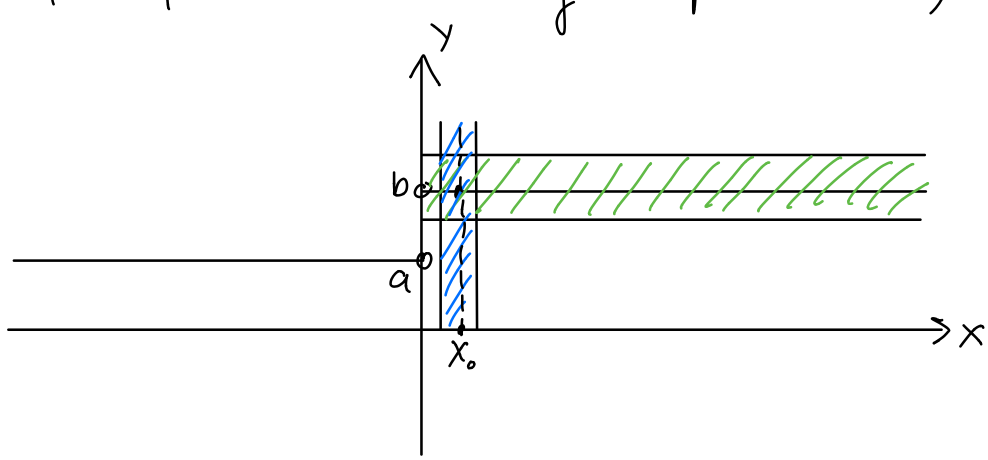
**Example 4.5:**

i) Is $f: D \to \mathbb{R}$ continuous, $U \subset D$, then the restricted function $f|_U : U \to \mathbb{R}$ is also continuous.

ii) According to Example 4.3 iv) the piecewise constant function

$$g = a\chi_{(-\infty,0)} + b\chi_{(0,\infty)} : \mathbb{R}\setminus\{0\} \to \mathbb{R}$$

is continuous (on $\mathbb{R}\setminus\{0\}$; the domain of definition is very important!)



## Remark 4.2:

The last example demonstrates an important aspect of continuity. The function $g$ is continuous although the graph of $g$ makes a jump at $x_0 = 0$: The "jump point" $x_0 = 0$ does <u>not</u> belong to the domain of definition of $g$.

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For monotonic functions, we have the following

## Proposition 4.2:

Let $-\infty \leq a < b \leq \infty$, and $f : (a,b) \longrightarrow \mathbb{R}$ be "monotonically increasing", that is,

$$\forall \ x, y \in (a,b) : x \leq y \Rightarrow f(x) \leq f(y).$$

Then we have for each $x_0 \in (a,b)$ the "left- and right-sided limits"

$$f(x_0^+) := \lim_{x \to x_0, \, x > x_0} f(x), \quad f(x_0^-) := \lim_{\substack{x \to x_0, \\ x < x_0}} f(x),$$

and $f$ is continuous at $x_0$ if and only if $f(x_0^-) = f(x_0^+) = f(x_0)$.

Analogously, if it is monotonically decreasing.

Proof:

Let $x_0 \in (a,b)$. If $(x_k)_{k\in\mathbb{N}} \subset (a,b)$ with

$$x_k < x_{k+1} \longrightarrow x_0 \ (k \to \infty),$$

then the sequence $(f(x_k))_{k\in\mathbb{N}}$ is monotonically increasing and bounded. Then

$$\text{Prop. 3.8} \Rightarrow \lim_{k\to\infty} f(x_k) := s \text{ exists}$$

We show that the limit is independent from the chosen sequence.

Claim: $\quad s = \lim_{\substack{x \to x_0, \\ x < x_0}} f(x) = f(x_0^-)$

Proof:

Let $(y_k)_{k\in\mathbb{N}} \subset (a,b)$ with $y_k \to x_0 \ (k \to \infty)$, where $y_k < x_0$, $k \in \mathbb{N}$. For $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ s.th.

$$\forall \ k \geq k_0 : \ s - \varepsilon < f(x_k) \leq s$$

As $x_{k_0} < x_0$, $y_k \to x_0 \ (k \to \infty)$, there is $k_1 \in \mathbb{N}$ s.th. $\quad \forall \ k \geq k_1 : \ x_{k_0} < y_k < x_0$

Together with the monotomy of $f$ we then get

$$\forall \quad K \geq K_1 \; : \; S - \varepsilon < f(x_{K_0}) \leq f(y_K) \leq \lim_{K \to \infty} f(x_K) = S;$$

That is,

$$f(y_K) \longrightarrow S \quad (K \to \infty)$$

Analogously, $f(x_0^+)$ exists. Aparently, we have

$$\lim_{x \to x_0} f(x) = f(x_0) \iff f(x_0^-) = f(x_0^+) = f(x_0)$$

$\square$

<u>Proposition 4.3:</u>

Let $f : \Omega \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then their "composition" $g \circ f : \Omega \to \mathbb{R}$ is also continuous.

<u>Proof:</u>

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\Omega$ and $\lim_{n \to \infty} x_n = a$.

As $f$ is continuous at $a$ we have

$$\lim_{n \to \infty} f(x_n) = f(a)$$

Then $y_n := f(x_n) \in \mathbb{R}$ and $\lim_{n \to \infty} y_n = b$.

As $g$ is continuous at $y = b$, we have

$$\lim_{n \to \infty} g(y_n) = g(b)$$

and therefore

$$\lim_{n \to \infty} (g \circ f)(x_n) = \lim_{n \to \infty} g(f(x_n)) = \lim_{n \to \infty} g(y_n) = b$$

$$= g(f(a)) = (g \circ f)(a)$$

$\square$

Proposition 4.4:

Let $f, g: \Omega \to \mathbb{R}$ be continuous functions and let $\lambda \in \mathbb{R}$. Then the functions

$f+g: \Omega \to \mathbb{R}$, $\lambda f: \Omega \to \mathbb{R}$, $fg: \Omega \to \mathbb{R}$

are also continuous. Furthermore, $\frac{f}{g}: \Omega' \to \mathbb{R}$ is continuous for $\Omega' = \{x \in \Omega \mid g(x) \neq 0\}$

Proof:

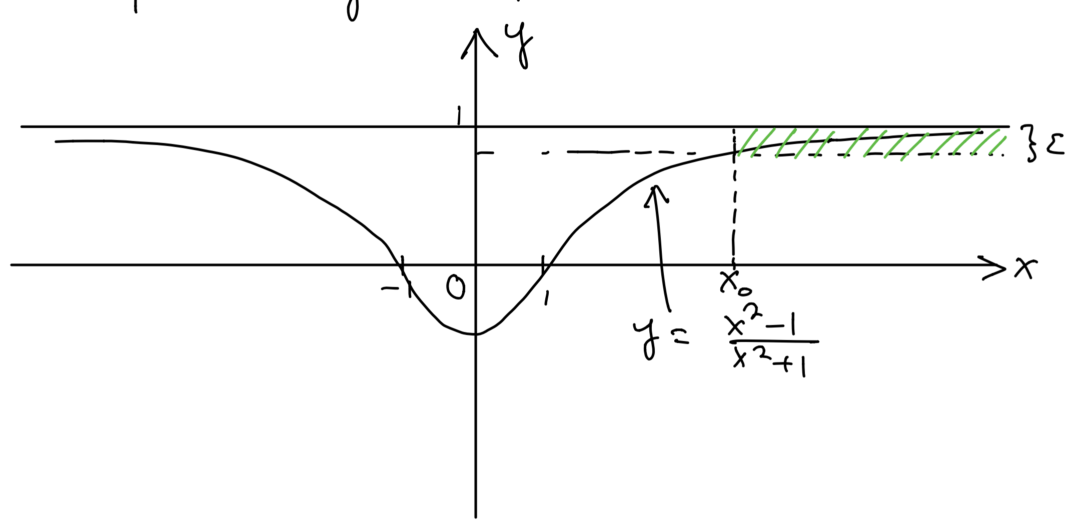
Follows from Prop. 3.3 $\square$

## §4.4 Limits at infinity and infinite limits

Let's look at the following function

$$f(x) = \frac{x^2-1}{x^2+1}$$

as $x$ becomes large. The graph of $f$ has the following shape



As $x$ gets larger, $f(x)$ approaches the value 1. Indeed, we have

$$\lim_{x\to\infty}\left(\frac{x^2-1}{x^2+1}\right) = \lim_{x\to\infty}\left(\frac{\cancel{x^2}(1-1/x^2)}{\cancel{x^2}(1+1/x^2)}\right)$$

and $\left|\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} - 1\right| < \varepsilon$ is equivalent to

$$\left|1-\frac{1}{x^2} - 1 - \frac{1}{x^2}\right| < \varepsilon\left(1+\frac{1}{x^2}\right) = \tilde{\varepsilon} > \varepsilon$$

Therefore, we have to show:

$$\exists\, x_0 \in \mathbb{R} \text{ s.t. } \forall\, x \geq x_0 : \frac{2}{x^2} < \varepsilon$$

Choose $x_0 = \sqrt{\frac{2}{\varepsilon}}$

This is symbolically written as

$$\lim_{x\to\infty}\frac{x^2-1}{x^2+1} = 1$$

In general, we use the notation

$$\lim_{x\to\infty} f(x) = L$$

to indicate that $f(x)$ approaches $L$ for large $x$.

<u>Definition 4.5:</u>

Let $f$ be a function defined on some interval $[a,\infty)$. Then

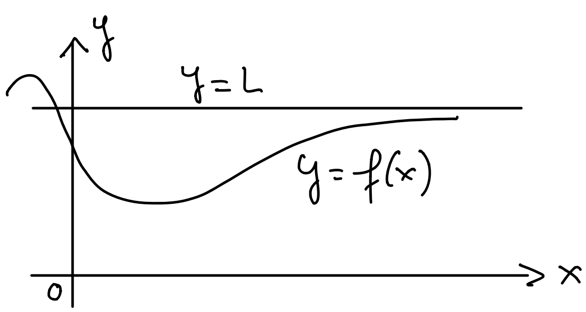
$$\lim_{x\to\infty} f(x) = L$$

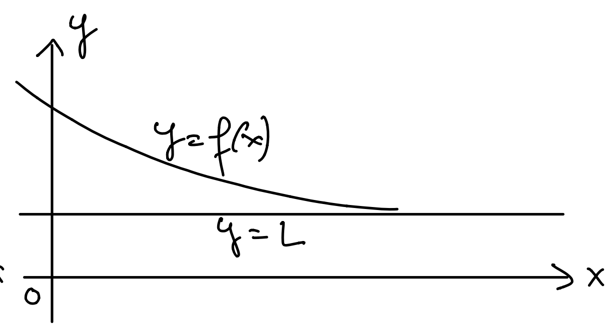
means that $\forall \ \varepsilon > 0 \ \exists \ x_0 \in \mathbb{R}$ s.t.

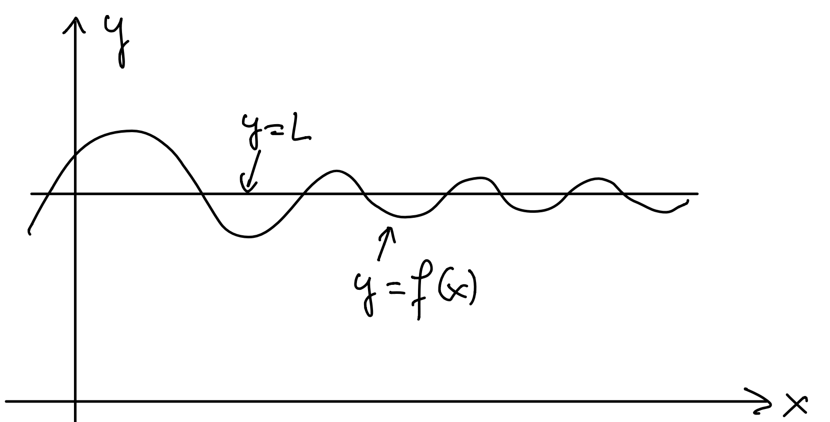
$$\forall \ x \geq x_0 : |L - f(x)| < \varepsilon$$

We say: "the limit of $f(x)$, as $x$ approaches infinity, is $L$"

There are many ways in which $L$ can be approached




Going back to our original function, we also approach a limit when $x$ decreases through negative values:

$$\lim_{x \to -\infty} \frac{x^2-1}{x^2+1} = 1$$

The general definition is analogous to 4.5

Definition 4.6:
The line $y = L$ is called "horizontal asymptote" of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

Example 4.6 (Infinite limits):

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists

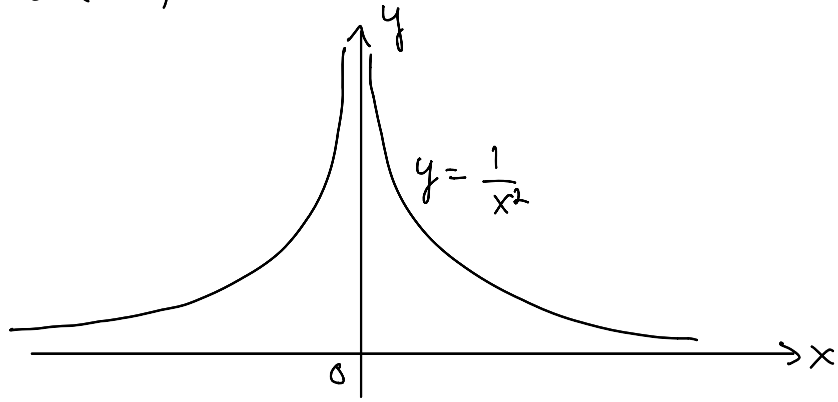
solution: consider the sequence

$$x_n = \frac{1}{n}, \quad n \in \mathbb{N}$$

Then $\forall\ M \in \mathbb{N}$, $M > 0$ $\exists$ $0 < N \in \mathbb{N}$ s.t.

$$\frac{1}{x_n^2} = \frac{1}{\left(\frac{1}{n}\right)^2} = n^2 > M \quad \forall\ n \geq N$$

Thus the function $\frac{1}{x^2}$ grows without bound as $x$ approaches $0$, so $\lim_{x \to 0} \left(\frac{1}{x^2}\right)$ does not exist.



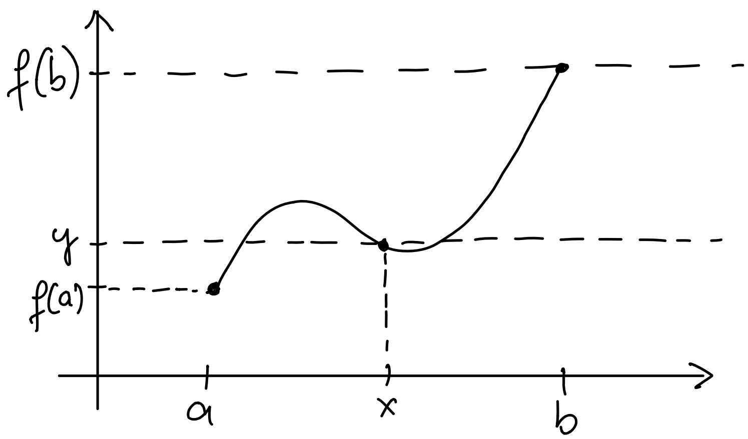

We use the notation: $\lim_{x \to 0} \frac{1}{x^2} = \infty$

The axis $x = 0$ is called "vertical asymptote".

## §4.5 Intermediate value theorem and applications

**Proposition 4.5:**

Let $-\infty < a < b < \infty$, and let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous, $f(a) \le f(b)$. Then for each $y \in [f(a), f(b)]$ there exists a $x \in [a,b]$ with $f(x) = y$




**Proof:**

Define $a_1 = a$, $b_1 = b$. set

$$a_2 = a = a_1, \quad b_2 = \frac{a+b}{2} \qquad \text{if } f\left(\tfrac{a+b}{2}\right) \ge y,$$

$$a_2 = \frac{a+b}{2}, \quad b_2 = b_1 \qquad \text{if } f\left(\tfrac{a+b}{2}\right) < y$$

such that $f(a_2) < y \le f(b_2)$ and $|a_2 - b_2| = \frac{|a-b|}{2}$

More generally, let $a_1, ---, a_K$ be already given s.t. $a_1 \leq \cdots \leq a_k \leq b_k \leq \cdots \leq b_1$

and $f(a_k) < y < f(b_k)$, $|a_k - b_k| = |a-b|\, 2^{1-k}$

Let $c = \dfrac{a_k + b_k}{2}$. If $f(c) \geq y$, set

$$a_{k+1} = a_k, \quad b_{k+1} = c,$$

if $f(c) < y$, set

$$a_{k+1} = c, \quad b_{k+1} = b_k$$

We obtain in each case $a_{k+1} \geq a_k$, $b_{k+1} \leq b_k$ with $f(a_{k+1}) < y \leq f(b_{k+1})$ and

$$|a_{k+1} - b_{k+1}| = \tfrac{1}{2}|a_k - b_k| = 2^{-k}|a-b|$$

The sequences $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ are monotonic and bounded. Prop. 3.8 then gives

$$\bar{a} = \lim_{k\to\infty} a_k \leq \bar{b} = \lim_{k\to\infty} b_k,$$

and Prop. 3.3 (sums of limits are limits of sums) then gives

$$|\bar{a} - \bar{b}| = \lim_{k\to\infty} |a_k - b_k| = 0$$

That is, $\bar{a} = \bar{b} =: x \in [a,b]$. As $f$ is continuous,

$$y \leq \lim_{k\to\infty} f(b_k) = f(x) = \lim_{k\to\infty} f(a_k) \leq y,$$

so $f(x) = y$.

$\square$

## Example 4.7:

i) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial of odd degree. Then $p$ has a zero.

Proof:

Observe that $|p(x)| \to \infty$ for $|x| \to \infty$.
Without loss of generality $p(x) \to \infty$ for $x \to \infty$.
(Otherwise consider $\tilde{p} = -p$.) As $p$ is of odd degree, we have $p(x) \to -\infty$ for $x \to -\infty$, and the claim follows from Prop. 4.5. $\square$

ii) Every $3 \times 3$ matrix $A$ with coefficients in $\mathbb{R}$ has at least one real eigenvalue

Proof:

The characteristic polynomial $p$ of $A$ is of degree 3, and the zeros of $p$ are exactly the eigenvalues of $A$:

$$Av = \lambda v \iff (A - \lambda \mathbb{1}) = 0$$

$$\Rightarrow p := \det(A - \lambda \mathbb{1}) = 0$$

is of degree 3

(see "Linear algebra") $\square$

Corollary 4.1:

Let $f: [a,b] \longrightarrow [a,b]$ be smooth. Then

$\exists\ x_0 \in [a,b]$ s.t. $f(x_0) = x_0$.

Proof:

Define the function $g(x) = x - f(x)$, $x \in [a,b]$.

Then $g: [a,b] \longrightarrow \mathbb{R}$ is smooth with

$$g(a) = a - f(a) \leq 0 \leq b - f(b) = g(b)$$

Prop. 4.5 then implies the existence of $x_0 \in [a,b]$ with $g(x_0) = 0 \iff f(x_0) = x_0$ $\square$